Week 8 Tuesday - Applied Optimization

Steps for optimization problems: (2 keys!)

① Find a formula☆ of what we what to optimize

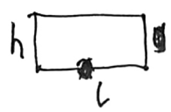
② Find the equation of constraint☆ in question (usually given by some numbers in question)

● Solve one variable in terms of another (Generally, solve the "easier" one)

③ Domain

④ Find global max/min & Justify by table of f'

1. *Above steps must be quite abstract. Let's look at one easier example.*
   Which rectangle of area 100 in^2 minimizes its height plus two times its length?



h　　　　L

Formula:

$S = h + 2l$

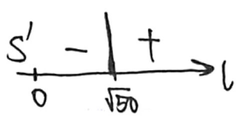
Constraint:

$hl = 100$

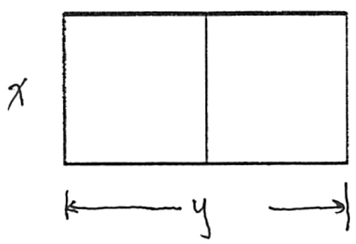
$\Rightarrow h = \frac{100}{l}$

$S = \frac{100}{l} + 2l \quad (l > 0)$

$S' = -\frac{100}{l^2} + 2 = 0 \Rightarrow$ critical pts: $l = \sqrt{50}$



2. A farmer wants to fence an area of 300 square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. Find the dimensions of the fenced area that will minimize the amount of the fencing material used.



Formula

$M = 3x + 2y$

Constraint

$xy = 300$

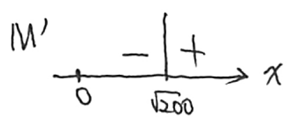
$\Rightarrow y = \frac{300}{x}$

$M = 3x + \frac{600}{x} \quad (x > 0)$

$M' = 3 - \frac{600}{x^2} = 0 \quad \Rightarrow 3 = \frac{600}{x^2} \Rightarrow x^2 = \frac{600}{3} = 200$

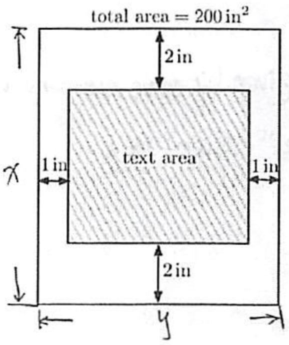
critical pts: $x = \sqrt{200}$

$y = \frac{300}{\sqrt{200}}$



When height is $\sqrt{200}$ ft, width is $\frac{300}{\sqrt{200}}$ ft, material is minimized.

3. You are designing a rectangular poster that will have a total area of 200 in^2. It should have 2-inch margins at the top and bottom, and a 1-inch margin at each side. What is the largest text area that you can have, and what dimensions of the poster give the largest text area?



Formula

$T = (x-4)(y-2)$

Constraint

$xy = 200$

$y = \frac{200}{x}$

$T = (x-4)\left(\frac{200}{x} - 2\right)$

$= 200 - \frac{800}{x} - 2x + 8 = 208 - \frac{800}{x} - 2x \quad (x > 0)$

$T' = \frac{800}{x^2} - 2 = 0$

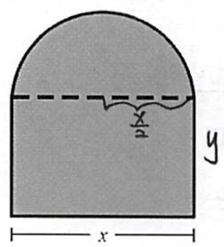
CP: $x = 20$

$y = \frac{200}{20} = 10$

T': $+$ | $-$ at $x = 20$

largest ~~text~~ area $T_{max} = (20-4)(10-2) = 16 \times 8 = 128$

4. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of such a window is 30 ft. Find the dimension of the window so that the greatest possible amount of light is admitted (that is, the total area is the largest).



Formula:

$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= xy + \frac{\pi}{8}x^2$

Constraint:

$\frac{2\pi \frac{x}{2}}{2} + x + 2y = 30$

$\left(\frac{\pi}{2}+1\right)x + 2y = 30$

$y = \frac{30 - \left(\frac{\pi}{2}+1\right)x}{2} > 0$

$\Downarrow$

$x < \frac{30}{\frac{\pi}{2}+1}$

$= x\,\frac{30 - \left(\frac{\pi}{2}+1\right)x}{2} + \frac{\pi}{8}x^2$

$= 15x - \frac{\pi}{4}x^2 - \frac{1}{2}x^2 + \frac{\pi}{8}x^2$

$= 15x - \left(\frac{\pi}{8} + \frac{1}{2}\right)x^2 \quad \left(0 < x < \frac{30}{\frac{\pi}{2}+1}\right)$

$A' = 15 - 2\left(\frac{\pi}{8}+\frac{1}{2}\right)x = 0$

$x = \frac{15}{2\left(\frac{\pi}{8}+\frac{1}{2}\right)} = \frac{15}{\frac{\pi}{4}+1}$

A': $+$ | $-$ on x: 0, $\frac{15}{\frac{\pi}{4}+1}$, $\frac{30}{\frac{\pi}{2}+1}$

$y = \frac{30 - \left(\frac{\pi}{2}+1\right)\frac{15}{\frac{\pi}{4}+1}}{2}$